Structured Markov chains solver: the algorithms

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ABSTRACT

We analyze the problem of the numerical solution of structured Markov chains encountered in queuing models: we describe the main computational problems and present the most advanced algorithms currently available for their solutions.

1. INTRODUCTION

We collect the most advanced algorithms for the numerical solution of structured Markov chains encountered in queuing problems, together with their main features and computational properties.

More specifically, we consider Quasi-Birth-Death processes (QBD), M/G/1 and G/M/1-type Markov chains, and non-skip-free processes (NSF). These Markov chains are defined by a semi-infinite transition matrix P, which is either a block tridiagonal matrix, or a (generalized) block Hessenberg matrix, and except for the first block row and the first block column, the blocks on each diagonal of P are constant. The latter property defines the class of block Toeplitz matrices. General references on these classes of problems can be found in the books [18, 19, 14, 4].

The specific structures of P and the related computational problems are described in Section 2. In Section 3 we present a general acceleration technique which can be used by all the algorithms. In Section 4 we describe the main features of the algorithms for solving the computational problems of Section 2. These algorithms are implemented in the software tool Structured Markov chain solver, which is described in the paper [8].

2. THE PROBLEMS

*This work has been partially supported by MIUR grant number 2004015437.
†B. Van Houdt is a postdoctoral fellow of the FWO-Flanders.

SMCTOOLS '06 Pisa, Italy

We consider row stochastic matrices P which can be partitioned into $m \times m$ blocks $P_{i,j}$. The matrix P is semi-infinite, i.e., its blocks $P_{i,j}$ have subscripts $i, j \in \mathbb{N}$. In this framework, the main computational problem is computing the *invariant probability vector*, i.e., the infinite nonnegative row vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} P = \boldsymbol{\pi}$, and $\boldsymbol{\pi}\boldsymbol{e} = 1$. Here $\boldsymbol{e}$ is the vector with all its entries equal to 1. Throughout the paper we assume P irreducible.

Due to the block structure of P, it is convenient to partition the vector $\boldsymbol{\pi}$ into subvectors $\pi_i \in \mathbb{R}^m$, $i \in \mathbb{N}$. According to the specific structure of P we may classify the Markov chains into suitable classes. In this section we refer the reader to the books [18, 19, 14, 4]. In the following section we restrict ourselves to the standard boundary behavior, the tool however also supports more general boundary conditions.

2.1 QBD Markov chains

QBD Markov chains are defined by the transition matrix

$$P = \begin{bmatrix} B_0 & A_1 & & & 0 \\ B_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & A_{-1} & A_0 & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix},$$

where $A_{-1}, A_0, A_1 \in \mathbb{R}^{m\times m}$ and B_0, $B_1 \in \mathbb{R}^{m\times m}$, are nonnegative matrices such that $A_{-1}+A_0+A_1$, $B_{-1}+A_0+A_1$ and B_0+A_1 are stochastic. Assume that $A = A_{-1}+A_0+A_1$ is irreducible. The drift of a QBD Markov chain is defined by $\mu = \boldsymbol{\alpha}^{\mathrm{T}}(-A_{-1}+A_1)\boldsymbol{e}$, where $\boldsymbol{\alpha}$ is the stationary probability vector of A. We recall that a QBD is positive recurrent if $\mu < 0$, null recurrent if $\mu = 0$ and transient if $\mu > 0$.

Define G, R and U the minimal nonnegative solutions of the matrix equations

$$\begin{aligned} G &= A_{-1} + A_0 G + A_1 G^2, \\ R &= A_1 + R A_0 + R^2 A_{-1}, \\ U &= A_0 + A_1 (I-U)^{-1} A_{-1}. \end{aligned} \tag{1}$$

One has $G = (I-U)^{-1}A_{-1}$, $R = A_1(I-U)^{-1}$, moreover, if the QBD is positive recurrent, it holds

$$\begin{aligned} \boldsymbol{\pi}_n &= \boldsymbol{\pi}_0 R^n, \qquad \text{for } n \ge 0 \\ \boldsymbol{\pi}_0(B_0 + R B_{-1}) &= \boldsymbol{\pi}_0 \\ \boldsymbol{\pi}_0 (I-R)^{-1}\boldsymbol{e} &= 1. \end{aligned}$$

2.2 M/G/1-type Markov chains

M/G/1-type Markov chains are defined by the transition matrix

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \dots \\ A_{-1} & A_0 & A_1 & A_2 & \dots \\ & A_{-1} & A_0 & A_1 & \ddots \\ & & A_{-1} & A_0 & \ddots \\ 0 & & & \ddots & \ddots \end{bmatrix},$$

where A_i, for $i \geq -1$, and B_i, for $i \geq 0$, are nonnegative matrices in $\mathbb{R}^{m\times m}$ such that $\sum_{i=-1}^{+\infty} A_i$, $\sum_{i=0}^{+\infty} B_i$, are stochastic. Throughout we assume $A = \sum_{i=-1}^{+\infty} A_i$ irreducible. The drift of an M/G/1-type Markov chain is defined by $\mu = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{a}$, where $\boldsymbol{\alpha}$ is the stationary probability vector of A and $\boldsymbol{a} = \sum_{i=-1}^{+\infty} iA_i\boldsymbol{e}$. We recall that a Markov chain is recurrent iff $\mu \leq 0$, positive recurrent iff $\mu < 0$ and $\sum_{i=1}^{+\infty} iB_i\boldsymbol{e} < \infty$, transient iff $\mu > 0$, null recurrent iff either $\mu = 0$ or $\mu < 0$ and $\sum_{i=-1}^{+\infty} iB_i\boldsymbol{e} = \infty$.

Define G the minimal nonnegative solution of the matrix equation

$$G = \sum_{i=-1}^{+\infty} A_i G^{i+1}. \tag{2}$$

If the Markov chain is positive recurrent, the following recursive formula due to Ramaswami [20] holds

$$\boldsymbol{\pi}_n = \left(\boldsymbol{\pi}_0 B_n^* + \sum_{i=1}^{n-1} \boldsymbol{\pi}_i A_{n-i}^*\right)(I - A_0^*)^{-1}, \qquad \text{for } n \geq 1, \tag{3}$$

where

$$A_n^* = \sum_{i=n}^{+\infty} A_i G^{i-n}, \qquad B_n^* = \sum_{i=n}^{+\infty} B_i G^{i-n}, \qquad \text{for } n \geq 0, \tag{4}$$

and $\boldsymbol{\pi}_0$ is such that

$$\boldsymbol{\pi}_0 B_0^* = \boldsymbol{\pi}_0, \quad \boldsymbol{\pi}_0\boldsymbol{b} - \mu\boldsymbol{\pi}_0\boldsymbol{e} + \boldsymbol{\pi}_0(I-B)(I-A)^{\#}\boldsymbol{a} = -\mu, \tag{5}$$

where $B = \sum_{n=0}^{+\infty} B_n$ and the operator $(\cdot)^{\#}$ denotes the group inverse.

A fast version of Ramaswami formula based on FFT is shown in [15]. It outperforms the formula based on (3) and (4) only if a very large number of components $\boldsymbol{\pi}_n$ is required.

2.3 G/M/1-type Markov chains

G/M/1-type Markov chains are defined by the transition matrix

$$P = \begin{bmatrix} B_0 & A_1 & & & 0 \\ B_{-1} & A_0 & A_1 & & \\ B_{-2} & A_{-1} & A_0 & A_1 & \\ B_{-3} & A_{-2} & A_{-1} & A_0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where A_{-i}, $i \geq -1$, and B_{-i}, $i \geq 0$ are nonnegative matrices in $\mathbb{R}^{m\times m}$ such that $\sum_{i=-1}^{n-1} A_{-i} + B_{-n}$ is stochastic for all $n \geq 0$.

If $A = \sum_{i=-1}^{+\infty} A_{-i}$ is not stochastic, then the Markov chain is positive recurrent. If A is stochastic then the Markov chain is positive recurrent if $\delta < 0$, null recurrent if $\delta = 0$, and transient if $\delta > 0$, where $\delta = \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{a}$, $\boldsymbol{\alpha}$ is such that $\boldsymbol{\alpha}^{\mathrm{T}}A = \boldsymbol{\alpha}^{\mathrm{T}}$, $\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{e} = 1$, and $\boldsymbol{a} = \sum_{i=-1}^{+\infty} iA_{-i}\boldsymbol{e}$.

Define R the minimal nonnegative solution of the matrix equation

$$R = \sum_{i=-1}^{+\infty} R^{i+1} A_{-i}. \tag{6}$$

If the Markov chain is positive recurrent, then

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R^n \qquad \text{for } n \geq 1, \tag{7}$$

where $\boldsymbol{\pi}_0$ is characterized by the system

$$\boldsymbol{\pi}_0 = \boldsymbol{\pi}_0 \sum_{i=0}^{+\infty} R^i B_{-i}, \qquad \boldsymbol{\pi}_0 (I - R)^{-1}\boldsymbol{e} = 1. \tag{8}$$

It is worth mentioning that, if the matrix $A = \sum_{i=-1}^{+\infty} A_{-i}$ is irreducible and stochastic, the connection between M/G/1 and G/M/1-type Markov chains is extremely simple. Indeed, define $D = \mathrm{diag}(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is the strictly positive stationary probability vector of A and define $\widetilde{A}_i = D^{-1}A_{-i}^{\mathrm{T}}D$, for $i = -1, 0, 1, \dots$. Now, take any solution R of (6) and define $\widetilde{G} = D^{-1}R^{\mathrm{T}}D$. It is easy to verify that $\widetilde{G}$ is a solution of

$$X = \sum_{i=-1}^{+\infty} \widetilde{A}_i X^{i+1}, \tag{9}$$

A consequence of this property is that, to determine R for a positive recurrent G/M/1-type Markov chain is equivalent to determining G for a transient M/G/1-type Markov chain. Conversely, to determine R for a null recurrent or transient G/M/1-type Markov chain is equivalent to determining G for a recurrent M/G/1-type Markov chain.

2.4 The case of Non-skip-free processes

Markov chains which are non-skip-free to lower levels are defined by the generalized block upper Hessenberg matrix

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \dots \\ B_{-1} & A_0 & A_1 & A_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ B_{-N+1} & A_{-N+2} & A_{-N+3} & A_{-N+4} & \ddots \\ A_{-N} & A_{-N+1} & A_{-N+2} & A_{-N+3} & \ddots \\ & A_{-N} & A_{-N+1} & A_{-N+2} & \ddots \\ & & A_{-N} & A_{-N+1} & \ddots \\ & & & A_{-N} & \ddots \\ 0 & & & & \ddots \end{bmatrix}. \tag{10}$$

for $m \times m$ blocks A_i, $i \geq -N$ and B_i, $i \geq -N+1$, where $N \geq 1$ is an integer. Markov chains which are non-skip-free to upper levels can be similarly defined in terms of generalized block lower Hessenberg matrix. The matrix P can be reblocked into blocks $\mathcal{B}_i$, $i \geq 0$ and $\mathcal{A}_i$, $i \geq -1$ of size mN as

$$P = \begin{bmatrix} \mathcal{B}_0 & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 & \dots \\ \mathcal{A}_{-1} & \mathcal{A}_0 & \mathcal{A}_1 & \mathcal{A}_2 & \dots \\ & \mathcal{A}_{-1} & \mathcal{A}_0 & \mathcal{A}_1 & \ddots \\ & & \mathcal{A}_{-1} & \mathcal{A}_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \quad (11)$$

Therefore, in principle, it can be solved like a standard M/G/1 Markov chain. In this case, the solution $\mathcal{G}$ of the equation $\mathcal{G} = \sum_{i=-1}^{+\infty} \mathcal{A}_i \mathcal{G}^{i+1}$ can be written as

$$\mathcal{G} = U^{-1} L \quad (12)$$

where

$$U = \begin{bmatrix} I & & & 0 \\ -G_1 & I & & \\ \vdots & \ddots & \ddots & \\ -G_{N-1} & \dots & -G_1 & I \end{bmatrix},$$

$$L = \begin{bmatrix} G_N & G_{N-1} & \dots & G_1 \\ & \ddots & \ddots & \vdots \\ & & G_N & G_{N-1} \\ 0 & & & G_N \end{bmatrix},$$

for suitable $m \times m$ matrices $G_1, \dots, G_N$. The matrix $\mathcal{G}$ can also be written as

$$\mathcal{G} = C(\boldsymbol{g})^N$$

where $\boldsymbol{g} = (G_N, G_{N-1}, \dots, G_1)$ and, given the block row vector $\boldsymbol{r} = (R_1, R_2, \dots, R_N)$ we define the block companion matrix associated with $\boldsymbol{r}$ as

$$C(\boldsymbol{r}) = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & I \\ R_1 & R_2 & \dots & \dots & R_N \end{bmatrix}. \quad (13)$$

3. SHIFT TECHNIQUES

For the sake of simplicity, assume that the matrix equation (2) can be rewritten as

$$G = \sum_{i=-1}^{M} A_i G^{i+1}. \quad (14)$$

In practice, this is the rule for the decay properties of the blocks A_i. According to the sign of the drift μ define the following blocks $\widetilde{A}_i$, $i \geq -1$. If $\mu \leq 0$ set

$$\widetilde{A}_{-1} = A_{-1}(I - Q),$$
$$\widetilde{A}_i = A_i - (\textstyle\sum_{j=-1}^{i} A_j - I)Q, \quad 0 \leq i \leq M,$$

where $Q = \boldsymbol{e}\boldsymbol{u}^T$ and $\boldsymbol{u}$ is any vector such that $\boldsymbol{e}^T\boldsymbol{u} = 1$. If $\mu > 0$ set

$$\widetilde{A}_{-1} = A_{-1}$$
$$\widetilde{A}_0 = A_0 + EA_{-1}$$
$$\widetilde{A}_i = A_i - E(I - \textstyle\sum_{j=-1}^{i-1} A_j), \quad 1 \leq i \leq M,$$

where $E = \boldsymbol{u}\boldsymbol{v}^T$, with $\boldsymbol{u}$ being any nonzero vector, and $\boldsymbol{v}$ such that $\boldsymbol{v}^T(\sum_{i=-1}^{M} A_i) = \boldsymbol{v}^T$, $\boldsymbol{v}^T\boldsymbol{u} = 1$.

Moreover, let us introduce the new equation

$$X = \sum_{i=-1}^{M} \widetilde{A}_i X^{i+1}. \quad (15)$$

It can be proved (see [4]) that the solution $\widetilde{G}$ of smallest spectral radius of (15) is

$$\widetilde{G} = G - Q \quad \text{if } \mu \leq 0,$$
$$\widetilde{G} = G \qquad \text{if } \mu > 0.$$

For QBD problems the new equation turns into

$$X = \widetilde{A}_{-1} + \widetilde{A}_0 X + \widetilde{A}_1 X^2 \quad (16)$$

where for $\mu \leq 0$ one has $\widetilde{A}_{-1} = A_{-1}(I-Q)$, $\widetilde{A}_0 = A_0 + A_1 Q$, $\widetilde{A}_1 = A_1$, whereas for $\mu > 0$ one has $\widetilde{A}_{-1} = A_{-1}$, $\widetilde{A}_0 = A_0 + EA_{-1}$, $\widetilde{A}_1 = (I - E)A_1$.

It has been proved that the roots of the polynomials $\widetilde{a}(z) = \det(\lambda I - \sum_{i=-1}^{M} z^{i+1}\widetilde{A}_i)$, and $a(z) = \det(\lambda I - \sum_{i=-1}^{M} z^{i+1} A_i)$, are the same except for the root $z = 1$ of $a(z)$ which is shifted to zero or to the infinity for $\widetilde{a}(z)$ according to the sign of the drift μ.

This tiny difference on the roots of $a(z)$ and $\widetilde{a}(z)$ makes a great difference in the convergence speed of the algorithms for the solution of matrix equations if applied to (14) or to (15). In fact, iterative methods converge faster if applied to equation (15). For more details on this shift technique we refer the reader to [10], [4], [2].

4. THE ALGORITHMS

In this section we discuss a number of algorithms for computing the minimal nonnegative solution G of the matrix equation (2) encountered in M/G/1 problems. Concerning the computation of the matrix R which solves (6) for G/M/1 problems, we rely on the reduction of G/M/1 to M/G/1 type Markov chains described in section 2.

4.1 Functional iterations

Methods based on functional iterations generate a sequence $\{X_k\}_k$ of matrices converging to the solution G once X_0 has been suitably chosen. We recall the three main iterations called natural, traditional and U-based.

$$X_{k+1} = \sum_{i=-1}^{+\infty} A_i X_k^{i+1} \qquad \text{natural}, \quad (17)$$

$$X_{k+1} = (I - A_0)^{-1}\Big(A_{-1} + \sum_{i=1}^{+\infty} A_i X_k^{i+1}\Big) \quad \text{traditional}, \quad (18)$$

$$X_{k+1} = \Big(I - \sum_{i=0}^{+\infty} A_i X_k^i\Big)^{-1} A_{-1} \qquad \text{U-based}. \quad (19)$$

If the initial matrix is $X_0 = 0$ then monotonic convergence of X_k to G occurs for all three sequences. Convergence is linear if $\mu \neq 0$, sublinear if $\mu = 0$. If the initial matrix is $X_0 = \sigma I$, where $\sigma = \rho(G)$ and $\rho(\cdot)$ denotes the spectral radius, then monotonicity is lost but the convergence is faster. We recall that if $\mu \leq 0$ then $\sigma = 1$, if $\mu > 0$ then σ is the smallest positive solution of the equation $z = \rho(A(z))$, where $A(z) = \sum_{i=-1}^{+\infty} z^{i+1} A_i$.

The three iterations above can be applied to the "shifted" equation (15). Local convergence is guaranteed but no analysis has been carried out concerning the convergence properties related to the choice of X_0. More details on functional iterations can be found in [4, 11, 16].

For non-skip-free Markov chains one has to compute the matrix $\mathcal{G}$ of (12). In this case it is sufficient to compute the blocks $G_1, \ldots, G_N$. These matrices can be approximated by means of the sequence of block row vectors $\boldsymbol{x}_n$, $n \geq 0$ generated from $\boldsymbol{x}_0$ by:

$$\boldsymbol{x}_{n+1} = [A_{-N}, A_{-N+1}, \ldots, A_{-1}] + \sum_{i=N}^{M+N} A_{i-N} \boldsymbol{x}_n C(\boldsymbol{x}_n)^{i-N}.$$

In fact, if $\boldsymbol{x}_0 = 0$ the sequence $\{\boldsymbol{x}_n\}$ monotonically converges to the first block row $[G_N, \ldots, G_1]$ of $\mathcal{G}$. The products $\boldsymbol{r}_j = \boldsymbol{x}_n C(\boldsymbol{x}_n)^j$ for $j = 0, \ldots, M$, are computed by means of the equation

$$\boldsymbol{r}_{j+1} = \boldsymbol{r}_j C(\boldsymbol{x}_n), \quad j = 0, 1, \ldots, M$$

starting with $\boldsymbol{r}_0 = \boldsymbol{x}_n$. This iteration has been introduced and analyzed in [9].

A faster convergence method is Newton's iteration [4], [12]. This method generates the sequence $X_{n+1} = X_n - W_n$, $n \geq 0$, $X_0 = 0$, where W_n solves the linear matrix equation

$$W_n - K_n \sum_{i=1}^{+\infty} A_i \sum_{j=0}^{i-1} X_n^j W_n X_n^{i-j-1} K_n A_{-1} = X_n - K_n A_{-1}$$

and $K_n = (I - \sum_{i=0}^{+\infty} A_i X_n^i)^{-1}$. Its convergence is quadratic if $\mu \neq 0$. The above matrix equation can be solved by means of $O(m^6)$ arithmetic operations. For QBD Markov chains the complexity reduces to $O(m^3)$.

4.2 Logarithmic reduction and cyclic reduction for QBD

Logarithmic reduction [13], [4] and cyclic reduction [5], [4] generate sequences of matrices converging quadratically to G provided that $\mu \neq 0$. In the latter case, convergence turns to linear.

If these iterations are applied to the shifted equation (15) then quadratic convergence still occurs if $\mu = 0$.

4.2.1 Logarithmic reduction

Logarithmic reduction is synthesized by the following equations:

$$\begin{aligned} B_{-1}^{(n+1)} &= (C^{(n)})^{-1} (B_{-1}^{(n)})^2, \\ B_1^{(n+1)} &= (C^{(n)})^{-1} (B_1^{(n)})^2, \quad n \geq 0, \end{aligned} \tag{20}$$

where

$$C^{(n)} = I - B_{-1}^{(n)} B_1^{(n)} - B_1^{(n)} B_{-1}^{(n)} \tag{21}$$

and

$$B_{-1}^{(0)} = (I - A_0)^{-1} A_{-1}, \quad B_1^{(0)} = (I - A_0)^{-1} A_1.$$

The sequence

$$G_n = B_{-1}^{(0)} + \sum_{i=1}^{n} \left(\prod_{j=0}^{i-1} B_1^{(j)} \right) B_{-1}^{(i)} \tag{22}$$

converges to G.

Logarithmic reduction can be applied to the shifted equation (16) by defining $B_{-1}^{(0)} = (I - \widetilde{A}_0)^{-1} \widetilde{A}_{-1}$, $B_1^{(0)} = (I - \widetilde{A}_0)^{-1} \widetilde{A}_1$. The sequence G_n of (22) converges to the solution $\widetilde{G}$ of (16).

4.2.2 Cyclic reduction

Cyclic reduction is synthesized by the following equations:

$$\begin{aligned} K^{(n)} &= (I - A_0^{(n)})^{-1}, \\ A_{-1}^{(n+1)} &= A_{-1}^{(n)} K^{(n)} A_{-1}^{(n)}, \\ A_0^{(n+1)} &= A_0^{(n)} + A_{-1}^{(n)} K^{(n)} A_1^{(n)} + A_1^{(n)} K^{(n)} A_{-1}^{(n)}, \\ A_1^{(n+1)} &= A_1^{(n)} K^{(n)} A_1^{(n)}, \\ \widehat{A}_0^{(n+1)} &= \widehat{A}_0^{(n)} + A_1^{(n)} K^{(n)} A_{-1}^{(n)}, \end{aligned} \tag{23}$$

for $n = 0, 1, \ldots$, and $\widehat{A}_0^{(0)} = A_0$, $A_i^{(0)} = A_i$, $i = -1, 0, 1$. The sequence

$$G_n = (I - \widehat{A}_0^{(n)})^{-1} A_{-1} \tag{24}$$

converges to G.

Cyclic reduction can be applied to the shifted equation (16) by defining $\widehat{A}_0^{(0)} = \widetilde{A}_0$, $A_i^{(0)} = \widetilde{A}_i$, $i = -1, 0, 1$. If $\mu \leq 0$ the sequence G_n of (24) converges to G. If $\mu > 0$ the sequence generated by $G_n = (I - \widehat{A}_0^{(n)})^{-1} \widetilde{A}_{-1}$ converges to G.

4.3 Cyclic reduction for M/G/1-type Markov chains

Cyclic reduction can be extended to M/G/1 Markov chains by means of a functional interpretation. Given a matrix power series $F(z) = \sum_{i=0}^{+\infty} z^i F_i$ define the matrix power series

$$F_{\text{even}}(z) = \frac{1}{2}(F(\sqrt{z}) + F(-\sqrt{z})) = \sum_{i=0}^{+\infty} z^i F_{2i},$$

$$F_{\text{odd}}(z) = \frac{1}{2\sqrt{z}}(F(\sqrt{z}) - F(-\sqrt{z})) = \sum_{i=0}^{+\infty} z^i F_{2i+1},$$

Let $A^{(0)}(z) = \sum_{i=-1}^{+\infty} z^{i+1} A_i$, $\widehat{A}^{(0)}(z) = \sum_{i=0}^{+\infty} z^i A_i$, and define

$$\begin{aligned} K^{(n)}(z) &= (I - A_{\text{odd}}^{(n)}(z))^{-1}, \\ A^{(n+1)}(z) &= z A_{\text{odd}}^{(n)}(z) + A_{\text{even}}^{(n)}(z) K^{(n)}(z) A_{\text{even}}^{(n)}(z), \\ \widehat{A}^{(n+1)}(z) &= \widehat{A}_{\text{even}}^{(n)}(z) + \widehat{A}_{\text{odd}}^{(n)}(z) K^{(n)}(z) A_{\text{even}}^{(n)}(z). \end{aligned} \tag{25}$$

Then the sequence

$$G_n = (I - \widehat{A}^{(n)}(0))^{-1} A_{-1} \tag{26}$$

converges to G. Convergence is quadratic if $\mu \neq 0$, is linear if $\mu = 0$.

Cyclic reduction can be applied to the shifted equation (15) by defining $A^{(0)}(z) = \sum_{i=-1}^{+\infty} z^{i+1}\widetilde{A}_i$, $\widehat{A}^{(0)}(z) = \sum_{i=0}^{+\infty} z^i \widetilde{A}_i$. If $\mu \le 0$ the sequence G_n of (26) converges to G. If $\mu > 0$ the sequence generated by $G_n = (I - \widehat{A}^{(n)}(0))^{-1}\widetilde{A}_{-1}$ converges to G. If cyclic reduction is applied to the shifted equation (15) then convergence is quadratic even if $\mu = 0$.

Concerning the different ways for implementing cyclic reduction we refer the reader to the book [4, Chapter 7]. One of the most efficient implementations relies on the technique of evaluation/interpolation where the computation of the coefficients of the matrix power series $A^{(n+1)}(z)$ is performed by means of a point-wise computation of the right-hand side of (25) at the qth roots of the unity, followed by an interpolation stage. Here q must be a sufficiently large integer such that $\sum_{i\ge q} ||A_i^{(n+1)}||$ is negligible for some matrix norm $||\cdot||$. Such an integer q exists due to the decay to zero of the coefficients $A_i^{(n+1)}$ and can be efficiently computed by means of a suitable technique of [6]. The method obtained in this way is called *point-wise cyclic reduction*, its efficiency relies on the use of FFT for performing the evaluation and the interpolation stages [6]. Its complexity is related to the number of interpolation points needed in the computation, or, equivalently, to the value of the numerical degrees of the matrix power series $A^{(n)}(z)$.

4.4 The Ramaswami reduction

The Ramaswami reduction [21], [4], [7] allows one to reduce an M/G/1-type Markov chain into a QBD process with blocks of infinite size. In this way, algorithms for solving a QBD can be adapted for solving an M/G/1-type Markov chain.

Define the matrices

$$\mathcal{A}_1 = \begin{bmatrix} 0 & & & 0 \\ I & 0 & & \\ & I & 0 & \\ & & \ddots & \ddots \\ 0 & & & \end{bmatrix}, \qquad (27)$$

$$\mathcal{A}_0 = \begin{bmatrix} A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (28)$$

and

$$\mathcal{A}_{-1} = \begin{bmatrix} A_{-1} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \qquad (29)$$

Then the matrix

$$\mathcal{G} = \begin{bmatrix} G & 0 & 0 & \cdots \\ G^2 & 0 & 0 & \cdots \\ G^3 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (30)$$

is the minimal nonnegative solution of the equation $\mathcal{G} = \mathcal{A}_{-1} + \mathcal{A}_0\mathcal{G} + \mathcal{A}_1\mathcal{G}^2$, where G is the minimal nonnegative solution of (1).

Logarithmic reduction and cyclic reduction can be applied for computing $\mathcal{G}$ in order to compute G. The shift technique of section 3 can be applied either to the original equation or to the infinite QBD obtained after the Ramaswami reduction.

4.5 Invariant subspaces method

The *invariant subspaces* method consists in approximating the minimal nonnegative solution G of the matrix equation (14) by approximating the left invariant subspace of a suitable block companion matrix. The method can be applied only if $\mu \ne 0$.

Define the matrix polynomial $H(t) = \sum_{i=0}^{N} t^i H_i$ as

$$H(t) = (1-t)^{N-1}(1+t)I - \sum_{i=0}^{N}(1-t)^{N-i}(1+t)^i A_{i-1},$$

and the matrices $\widehat{H}_i = H_N^{-1}H_i$, $i = 0, \dots, N-1$. Set $\boldsymbol{h} = -[\widehat{H}_0, \dots, \widehat{H}_{N-1}]$ and introduce the matrix

$$F = C(\boldsymbol{h}) + \mathrm{sign}(\mu)\boldsymbol{y}\boldsymbol{x}^T/(\boldsymbol{x}^T\boldsymbol{y}),$$

where

$$\boldsymbol{x}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_0^{\mathrm{T}}\widehat{H}_1 & \boldsymbol{x}_0^{\mathrm{T}}\widehat{H}_2 & \cdots & \boldsymbol{x}_0^{\mathrm{T}}\widehat{H}_{N-1} & \boldsymbol{x}_0^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and $\boldsymbol{x}_0$, $\boldsymbol{y}_0$ are two vectors such that $\boldsymbol{x}_0^{\mathrm{T}}\widehat{H}_0 = 0$, $\widehat{H}_0\boldsymbol{y}_0 = 0$.

Define the matrix sign $S = \mathrm{Sign}(F)$ given by $S = \lim_n S_n$, where S_n is the sequence generated by

$$\begin{cases} S_0 = F \\ S_{n+1} = \frac{1}{2}\left(S_n + S_n^{-1}\right), \quad n \ge 0. \end{cases} \qquad (31)$$

Then $I - S$ has rank m and by means of a rank revealing QR factorization of $I - S$ it is possibe to compute an $Nm \times m$ matrix T whose columns are a basis of the linear space spanned by the columns of $I - S$. Denoting T_1 and T_2 the submatrices of T made up by the rows $1, 2, \dots, m$ and $m + 1, m + 2, \dots, 2m$, it holds

$$G = (T_1 + T_2)(T_1 - T_2)^{-1}.$$

For positive recurrent Markov chains the convergence of S_n to S is quadratic.

The matrix sign S can be computed by means of the iteration (31) which can be stopped if $||S_n - S_{n+1}|| < \epsilon$ for some matrix norm $||\cdot||$ and for a given $\epsilon > 0$. A different approach is to compute S by means of the Schur decomposition of F.

Acceleration techniques have been devised based on the computation of some determinants. For more details and for the theory behind this technique we refer the reader to [4] and [1]. Comparisons between cyclic reduction and invariant subspace can be found in [17], [3].

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