GRAPH CONNECTIVITY

9 Elementary Properties

DEFINITION 9.1: A graph G is said to be connected if for every pair of vertices there is a path joining them. The maximal connected subgraphs are called components.

DEFINITION 9.2: The connectivity number $\kappa(G)$ is defined as the minimum number of vertices whose removal from G results in a disconnected graph or in the trivial graph (=a single vertex). A graph G is said to be k-connected if $\kappa(G) \geq k$.

Clearly, if G is k-connected then $|V(G)| \geq k+1$ and for $n, m > 2$, $\kappa(K_n) = n-1$, $\kappa(C_n) = 2$, $\kappa(P_n) = 1$ and $\kappa(K_{n,m}) = \min(m,n)$.

DEFINITION 9.3: The connectivity number $\lambda(G)$ is defined as the minimum number of edges whose removal from G results in a disconnected graph or in the trivial graph (=a single vertex). A graph G is said to be k-edge-connected if $\lambda(G) \geq k$.

THEOREM 9.1 (Whitney): Let G be an arbitrary graph, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof: Let v be a vertex with $d(v) = \delta(G)$, then removing all edges incident to v disconnects v from the other vertices of G. Therefore, $\lambda(G) \leq \delta(G)$. If $\lambda(G) = 0$ or 1, then $\kappa(G) = \lambda(G)$. On the other hand, if $\lambda(G) = k \geq 2$, let $x_1y_1, x_2y_2, \ldots, x_ky_k$ are be the edges whose removal causes G to be disconnected (where some of the x_i, resp. y_i , vertices might be identical). Denote V_1 and V_2 as the components of this disconnected graph. Then, either V_1 contains a vertex v different from $x_1, x_2, \ldots, x_k$, meaning that removing $x_1, \ldots, x_k$ causes v to be disconnected from V_2. Or, $V_1 = \{x_1, \ldots, x_k\}$, where

$|V_1| \leq k$ (some x_i's might be identical). Now, in this case, x_1 has at most k neighbors (being $|V_1|-1$ in V_1 and $k-(|V_1|-1)$ in V_2). Moreover, $\lambda(G)=k$, thus, $d(x_1) = k$ and the removal of the k neighbors of x_1 cause G to be disconnected.

Q.E.D.

Let G be a graph of order $n \geq k+1 \geq 2$. If G is not k-connected then there are two disjoint sets of vertices V_1 and V_2, with $|V_1| = n_1 \geq 1$, $|V_2| = n_2 \geq 1$ and $n_1+n_2+k-1 = n$ such that the vertices of V_i have a degree of at most $n_i - 1 + k - 1$, $i = 1, 2$. (Indeed, the $k-1$ vertices that are not in $|V_1| \cup |V_2|$ separate the sets V_1 and V_2).

Corollary 9.1 (Bondy (1969)): Let G be a graph with vertices $x_1, x_2, \ldots,$ x_n, $d(x_1) \leq d(x_2) \leq \ldots \leq d(x_n)$. Suppose for some k, $0 \leq k \leq n$, that $d(x_j) \geq j+k-1$, for $j = 1, 2, \ldots, n-1-d(x_{n-k+1})$, then G is k-connected.

Proof: Suppose that G is not k-connected. Then $\exists V_1, V_2 \subset V(G)$ such that $V_1 \cap V_2 = \emptyset$, $|V_1| = n_1$, $|V_2| = n_2$, $n_1 + n_2 = n-k+1$ and $d(x) \leq n_i + k - 2$ for $x \in V_i$. Now, $X = \{x_j | j \geq n-k+1\}$ is a set of k elements all with a degree larger than or equal to $d(x_{n-k+1})$. Hence, there is at least one $x \in X \cap (V_1 \cup V_2)$. Without loss of generality, say in $X \cap V_2$.
Thus, $n_2 \geq d(x_{n-k+1})+1-(k-1) = d(x_{n-k+1})-k+2$ and $n_1 = n-k+1-n_2 \leq n-1-d(x_{n-k+1})$. Take $x_j \in V_1$ such that j is maximal ($j \geq n_1$), then $n_1+k-1 \leq d(x_{n_1}) \leq d(x_j) \leq n_1+k-2$ (by construction).

Q.E.D.

Thus, if G is a graph with vertices $x_1, x_2, \ldots, x_n$, with $d(x_1) \leq \ldots \leq d(x_n) = \Delta(G)$ and $d(x_j) \geq j$ for $j = 1, \ldots, n-\Delta(G)-1$, then G is connected. The reverse is, obviously, not true.

Corollary 9.2 (Chartrand and Harary (1968)): Let $G \neq K_n$ be a graph of order n, then $\kappa(G) \geq 2\delta(G) + 2 - n$.

Proof: Let $k = 2\delta(G)+2-n$. It suffices to show $d(x_j) \geq j+k-1$, for $j = 1, \ldots, n-1-\delta(G)$ (because $d(x_{n-k+1}) \geq \delta(G)$). This is certainly true if $d(x_j) \geq n-1-\delta(G)+k-1$ for all $j = 1, \ldots, n-1-\delta(G)$ and $n-1-\delta(G)+k-1 = \delta(G)$.

Q.E.D.

Exercises 9.1: On graph connectivity:

1. Give 4 graphs G_1, G_2, G_3 and G_4 such that $0 < \kappa(G_1) = \lambda(G_1) = \delta(G_1)$, $0 < \kappa(G_2) < \lambda(G_2) = \delta(G_2)$, $0 < \kappa(G_3) = \lambda(G_3) < \delta(G_3)$, and $0 < \kappa(G_4) < \lambda(G_4) < \delta(G_4)$.

2. Give a graph G such that $\kappa(G) = 2\delta(G) + 2 - n > 0$.

3. Determine the minimum $e(n)$ such that all graphs with n vertices and $e(n)$ edges are connected (= 1-connected).

4. Let G be a graph with n vertices and e edges, show $\kappa(G) \leq \lambda(G) \leq \lfloor 2e/n \rfloor$.

5. Let G be a graph with $\delta(G) \geq \lfloor n/2 \rfloor$, then G connected. Moreover, $\lambda(G) = \delta(G)$ [Hint: Prove that any component C_i of G, after removing $\lambda(G) < \delta(G)$ edges, contains at least $\delta(G) + 1$ vertices.].

6. Let G be any 3-regular graph, i.e., $\delta(G) = \Delta(G) = 3$, then $\kappa(G) = \lambda(G)$. Draw a 4-regular planar graph G such that $\kappa(G) \neq \lambda(G)$.

THEOREM 9.2: Given the integers n, δ, κ and λ, there is a graph G of order n such that $\delta(G) = \delta, \kappa(G) = \kappa$, and $\lambda(G) = \lambda$ if and only if one of the following conditions is satisfied:

1. $0 \leq \kappa \leq \lambda \leq \delta < \lfloor n/2 \rfloor$,

2. $1 \leq 2\delta + 2 - n \leq \kappa \leq \lambda = \delta < n - 1$,

3. $\kappa = \lambda = \delta = n - 1$.

Of course, if $\kappa(G) = 0$, then so is $\lambda(G)$.

Proof: Let G be any graph of order n with $\delta(G) = \delta, \kappa(G) = \kappa$, and $\lambda(G) = \lambda$. Then, (a) $\delta(G) < \lfloor n/2 \rfloor$, that is, condition 1 is true, or (b) $\lfloor n/2 \rfloor \leq \delta(G) < n-1$, meaning that $2\delta \geq 2\lfloor n/2 \rfloor \geq n-1$, or $2\delta + 2 - n \geq 1$. Thus, by Corollary 1.2 and Exercise 1.1.5 we have condition 2. Or (c) if $\delta(G) = n - 1$, then $G = K_n$ and $\kappa(G) = \lambda(G) = \delta(G) = n - 1$.
Thus we have to show that if condition $1, 2$ or 3 is satisfied then there is a graph G with appropriate constants $n, \kappa, \lambda, \delta$. Suppose that condition 1 holds. Let $G_1 = K_{\delta+1}$, $G_2 = K_{n-\delta-1}$, $u_1, \ldots, u_{\delta+1} \in G_1$ and $v_1, \ldots, v_{\delta+1} \in G_2$ (notice, $K_{\delta+1} \subset G_2$). Next, set $G = G_1 \cup G_2$ and add the edges $u_1v_1, \ldots, u_\kappa v_\kappa$ and $u_{\kappa+1}v_1, \ldots, u_\lambda v_1$ to G. Then, $\kappa(G) = \kappa$, by removing the vertices $v_1, \ldots, v_\kappa$,

$\lambda(G) = \lambda$, by removing the edges between G_1 and G_2, and $\delta(G) = \delta$, by considering the vertex $u_{\delta+1}$. Suppose that condition 2 holds. Let $G_1 = K_\kappa$, $G_2 = K_a$, $G_3 = K_b$ and $G_0 = G_1 + (G_2 \cup G_3)$, where $a = \lfloor (n-\kappa)/2 \rfloor$ and $b = \lfloor (n-1-\kappa)/2 \rfloor$ (notice, $a+b = n-\kappa-1$)[7]. To construct G, add a vertex v to G_0 and joint it to the vertices of G_1 and to $\delta - \kappa$ vertices of G_3 (this is possible because $2\delta + 2 - n \leq \kappa$ implies that $\delta - \kappa \leq b$). Then, $\kappa(G) = \kappa$, by removing the vertices of G_1, $\lambda(G) = \lambda$, by removing the edges to v, and $\delta(G) = \delta$, by considering the vertex v. Finally, if condition 3, holds, set $G = K_n$.

Q.E.D.

10 Menger's Theorem

DEFINITION 10.1: The local connectivity $\kappa(x, y)$ of two non-adjacent vertices is the minimum number of vertices separating x from y. If x and y are adjacent vertices, their local connectivity is defined as $\kappa_H(x, y) + 1$ where $H = G - xy$. Similarly, we define the local edge-connectivity $\lambda(x, y)$.

Clearly, $\kappa(G) = \min\{\kappa(x, y) | x, y \in G, x \neq y\}$. The aim of this section is to discuss the fundamental connections between $\kappa(x, y)$ and the set of xy paths. Two paths in a graph G are said to be independent if every common vertex is an endvertex of both paths. A set of independent xy paths is a set of paths any two of which are independent. Obviously, if there are k independent xy paths then $\kappa(x, y) \geq k$. Menger's Theorem states that the converse is true. We prove the theorem by means of an elegant proof by Dirac (1969).

THEOREM 10.1 (Menger (1926)): Let $x, y \in G, x \neq y$. There exists a set of $\kappa(x, y)$ independent paths between x and y and this set is maximal.

Proof: We use induction on $m = n + e$, the sum of the number of vertices and edges in G. We show that if $S = \{w_1, w_2, \ldots, w_k\}$ is a minimum set (that is, a subset of the smallest size) that separates x and y, then G has at least k independent paths between x and y. The case for $k = 1$ is clear, and this takes care of the small values of m, required for the induction.
(1) Assume that x and y have a common neighbor $z \in \Gamma(x) \cap \Gamma(y)$. Then necessarily $z \in S$. In the smaller graph $G - z$ the set $S - z$ is a minimum

[7] $G + H$ is used here to reflect the graph obtained by $G \cup H$ and adding an edge between every vertex $x \in G$ and $y \in H$

set that separates x and y, and so the induction hypothesis yields that there are $k-1$ independent paths between x and y in $G-z$. Together with the path xzy, there are k independent paths in G as required.
(2) Assume that $\Gamma(x) \cap \Gamma(y) = \emptyset$ and denote by H_x and H_y as the connected components of $G-S$ for x and y, respectively.
(2a) Suppose that the separating set $S \not\subset \Gamma(x)$ and $S \not\subset \Gamma(y)$. Let z be a new vertex, and define G_z to be the graph with the vertices $V(H_x \cup S \cup z)$ having the edges of $G[H_x \cup S]$ together with the edges zw_i for all $i = 1, \ldots, k$. The graph G_z is connected and it is smaller than G. Indeed, in order for S to be a minimum separating set, all w_i vertices have to be adjacent to some vertex in H_y. This shows that $e(G_z) \leq e(G)$ and, moreover, assumption (2a) rules out the case $H_y = y$, therefore $n(G_z) < n(G)$ in the present case. If T is any set that separates x and z in G_z, then T will separate x from all $w_i \in S - (T \cap S)$ in G. This means that T separates x and y in G. Since k is the size of a minimum separating set, $|T| = k$. We noted that G_z is smaller than G, and thus by the induction hypothesis, there are k independent paths from x to z in G_z. This is possible only if there exist k independent paths from x to w_i, for $i = 1, \ldots, k$, in H_x. Using a symmetric argument one finds k independent paths from y to w_i in H_y. Combining these paths proves the theorem.
(2b) Suppose that all separating sets S are a subset of $\Gamma(x)$ or $\Gamma(y)$. Let P be the shortest path from x to y in G, then P contains at least 4 vertices, we refer to the second and third node as u and v. Define G_n as $G - uv$ (that is, remove the edge between u and v). If the smallest set T that separates x from y in G_n has a size k, then by induction, we are done. Suppose that $|T| < k$, then x and y are still connected in $G - T$ and every path from x to y in $G - T$ necessarily travels along the edge uv. Therefore, $u, v \notin T$. Also, $T_u = T \cup u$ and $T_v = T \cup v$ are both minimum separating sets in G (of size k). Thus, $T_v \subset \Gamma(x)$ or $T_v \subset \Gamma(y)$ (by (2b)). Now, P is the shortest path, so $v \notin \Gamma(x)$, hence, $T_v \subset \Gamma(y)$. Moreover, $u \in \Gamma(x)$, thus $T_u \subset \Gamma(x)$. Combining these two results we find $T \subset \Gamma(x) \cap \Gamma(y)$ (and T is not empty). Which contradicts assumption (2).
The set is maximal, because the existence of k independent paths between x and y implies that $\kappa(x, y) \geq k$.

Q.E.D.

Another way to state this result is the following: A necessary and sufficient condition for a graph to be k-connected is that any two distinct vertices x and y can be joined by k independent paths.

EXERCISES 10.1: On Menger's Theorem:

- Let G be a graph with $|G| \geq k+1$, then G is k-connected if and only if for all k-element subsets $V_1, V_2 \in V(G)$, there is a set of k paths from V_1 to V_2 which have no vertex in common. [$V_1 \cap V_2$ is not necessarily empty, thus, some paths might be trivial paths].

Let U be a set of vertices of a graph G and let x be a vertex not in U. An xU fan is defined as a set of $|U|$ paths from x to U, any two of which have only the vertex x in common.

THEOREM 10.2 (Dirac (1960)): A graph G is k-connected if and only if $|G| \geq k+1$ and for any k-set $U \in V(G)$ and $x \in V(G) - U$, there is an xU fan.

Proof: (a) Suppose that G is k-connected, $U \subset V(G)$, $|U| = k$ and $x \in V(G) - U$. Let H be the graph obtained from G by adding a vertex y and joining y to every vertex in U. Clearly, H is also k-connected. Therefore, by Menger's theorem, we find that there are k independent xy paths in H. Omitting the edges incident with y, we find the required xU fan.
(b) Suppose $|G| \geq k+1$ and that S is a $(k-1)$-set separating x and y, for some vertices x and y. Then, G does not contain an $x(S \cup y)$ fan. Q.E.D.

EXERCISES 10.2: On Dirac's Theorem:

1. If G is k-connected ($k \geq 2$), then for any set of k vertices $\{a_1, \ldots, a_k\}$ there is a cycle containing all of them. [Hint: Use induction on k and distinguish between the case where the cycle C contains an additional vertex $x \notin \{a_1, \ldots, a_{k-1}\}$ and the case where it does not.]

2. Give an example of a graph for which there is for any set of k points, a cycle containing all of them, but that is not k-connected ($k > 2$).

11 Additional Exercises

DEFINITION 11.1: Let $k \geq 1$. Consider the set B^k of all binary sequences of length k. For instance, $B^3 = \{000; 001; 010; 100; 011; 101; 110; 111\}$. Let Q_k be the graph (called the k-cube) with $V(Q_k) = B^k$, where $uv \in E(Q_k)$ if and only if the sequences u and v differ in exactly one place.

EXERCISES 11.1: On k-cube graphs Q_k:

- Determine the order of Q_k. Show that Q_k is regular, and determine $e(Q_k)$ for each $k \geq 1$.
- Compute $\chi(Q_k)$ for all $k \geq 1$.
- Prove that $\kappa(Q_k) = \lambda(Q_k) = \delta(Q_k) = k$ [Hint: use induction on k. Consider the graphs G_0 and G_1 induced by the vertices $0u$ and $1u$, respectively. Let S be a minimal set that disconnects Q_k, then S must disconnect G_0 or G_1].
- Determine the k values for which Q_k is planar.